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A TWO-DIMENSIONAL SHEAR DEFORMABLE BEAM FOR LARGE ROTATION AND DEFORMATION PROBLEMS

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1. INTRODUCTION

In Euler–Bernoulli beam theory it is assumed that the beam cross-section remains rigid and perpendicular to the neutral axis of the beam [1, 2]. In this theory, the effect of the shear deformation is neglected. In Timoshenko beam theory on the other hand, the cross-section does not remain perpendicular to the beam neutral axis. Nonetheless, the cross-section remains rigid in many models. A shear coefficient is introduced in order to account for the shear deformation [3]. In this investigation, a two-dimensional shear-deformable beam element based on the non-incremental absolute nodal co-ordinate formulation [4] is developed. In this approach, only absolute co-ordinates and global slopes are used to define the element nodal co-ordinates without the need for using infinitesimal or finite rotations. Using this co-ordinate representation with the appropriate element shape function, exact modelling of the rigid body dynamics can be achieved. Using the non-incremental absolute nodal co-ordinate formulation, the resulting mass matrix of the finite element is a constant matrix and the centrifugal and Coriolis forces are identically equal to zero.

A problem encountered in the implementation of the non-incremental absolute nodal co-ordinate formulation is the formulation of the elastic forces. Shabana *et al.* [4–8] proposed two methods for formulating the elastic forces of the two-dimensional beam element. In the first method, a local element co-ordinate system is introduced for the convenience of describing the element deformation. This approach leads to a complex expression for the elastic forces even when a linear elastic model is used. In the second method [9] a continuum mechanics approach is used to obtain the elastic forces without introducing the local element co-ordinate system. In this continuum mechanics approach, non-linear strain–displacement relationships must be used in order to obtain zero strain under an arbitrary rigid body motion. Nonetheless, the previous models developed using the continuum mechanics approach are based on Euler–Bernoulli beam theory which does not account for the shear deformation. It is the objective of this investigation to develop a model for the elastic forces for two-dimensional beam elements that accounts for shear deformation by using a general continuum mechanics approach without introducing a local element co-ordinate system. This new model relaxes the assumptions of Euler–Bernoulli and Timoshenko beam models and does not require the use of a shear coefficient.

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2. KINEMATICS OF THE SHEAR-DEFORMABLE BEAM

For the two-dimensional shear-deformable beam element, the displacement field is defined in the global co-ordinate system as

$$\mathbf{r} = \begin{bmatrix} a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 x^3 \\ b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 x^3 \end{bmatrix}, \tag{1}$$

where $\mathbf{r}$, as shown in Figure 1, is the global position vector of an arbitrary point P in the beam cross-section, a_i and b_i are the polynomial coefficients, and x and y are the spatial co-ordinates defined in a beam co-ordinate system. The spatial co-ordinate x is chosen to be along the beam axis ($0 \leqslant x \leqslant l$), where l is the element length. Note that the assumed displacement field depends on y in order to account for the shear deformation.

In Euler–Bernoulli beam theory, the effect of the shear deformation is neglected [1]. The basic assumption in Euler–Bernoulli beam theory is that the cross-section of the beam remains normal to the beam neutral axis as shown in Figure 2(a). The beam cross-section at any point along the beam neutral axis can be defined by the Frenet frame. The Frenet frame has one of its axes tangent to the beam neutral axis and the other axis perpendicular to the

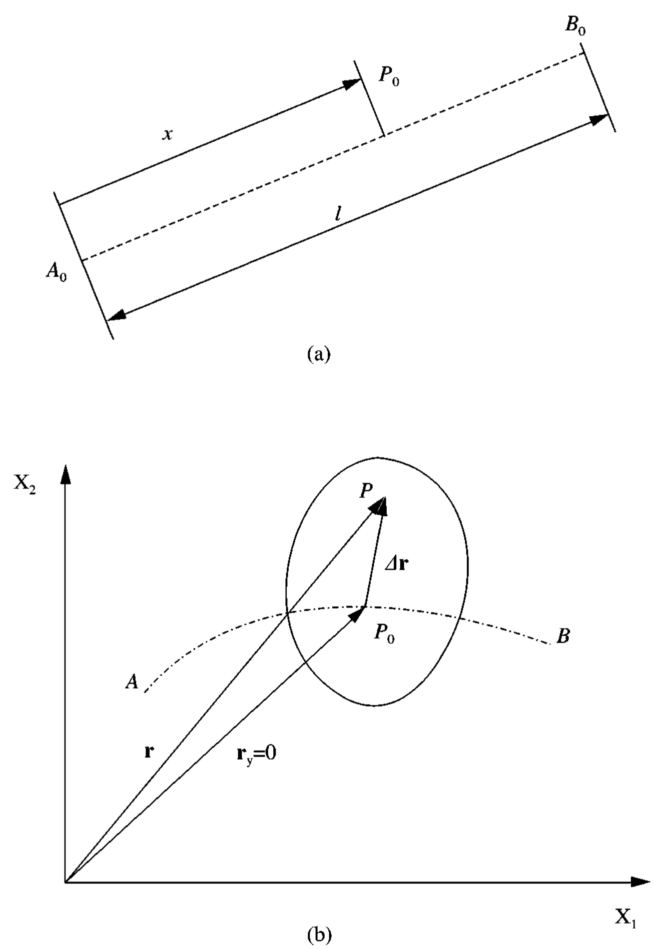

Figure 1. The position vector of an arbitrary point on the beam cross-section: (a) the beam in the undeformed configuration; (b) the beam in the deformed configuration.

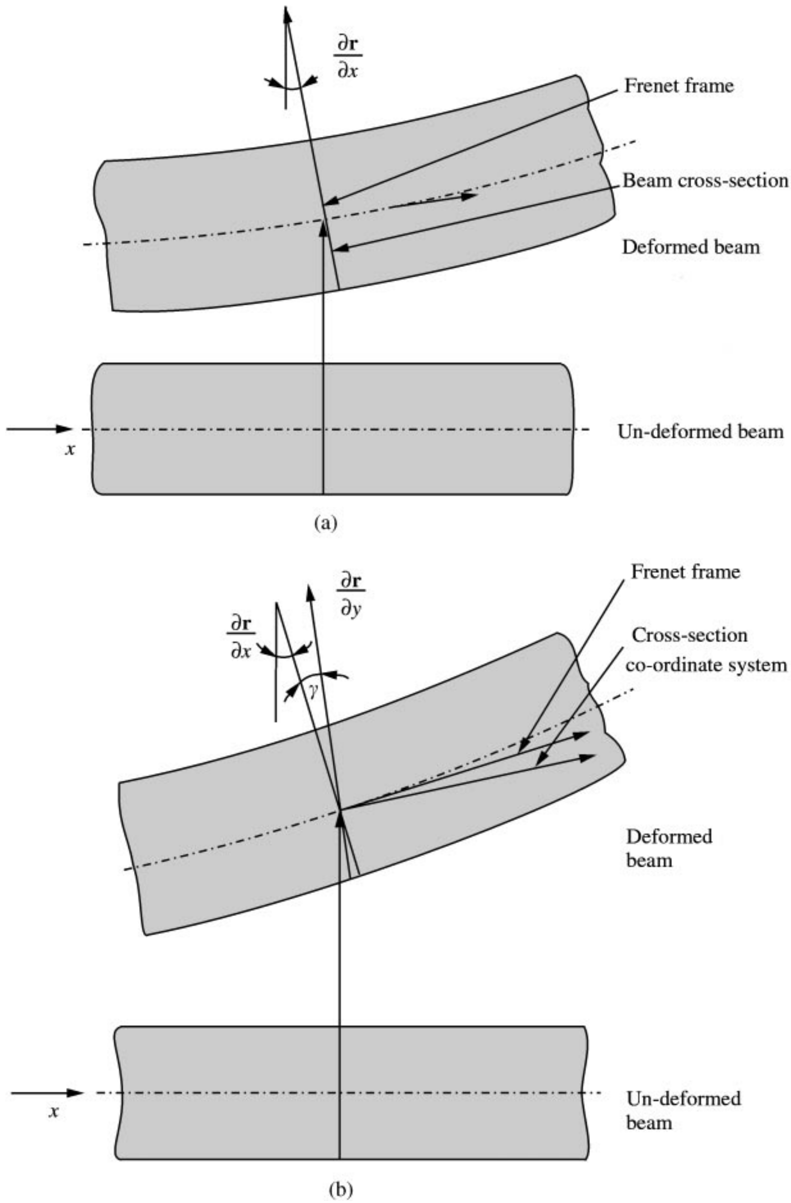

Figure 2. The beam deformation assumptions: (a) the Euler–Bernoulli beam without shear deformation; (b) the beam element with shear deformation.

beam neutral axis [10]. The tangent vector **t** can be defined by $\partial \mathbf{r}/\partial x$. In a shear-deformable beam model, the cross-section of the beam does not remain normal to the neutral axis, as shown in Figure 2(b). As a result, the tangent to the neutral axis cannot be used to define the cross-section. In order to demonstrate that the shape function of equation (1) accounts for the shear effect, consider an arbitrary vector $\Delta\mathbf{r}$, which is defined in the beam cross-section as shown in Figure 1. Using the displacement field defined by equation (1), it can be shown that

$$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_{y=0} = y\frac{\partial \mathbf{r}}{\partial y}, \tag{2}$$

where $\mathbf{r}$ is the global position vector of an arbitrary point P in the cross-section with co-ordinates (x, y), and $\mathbf{r}_{y=0}$ is the position vector of the corresponding point P_o on the beam centerline with co-ordinates $(x, 0)$. The preceding equation shows that any arbitrary vector drawn on the beam cross-section can be defined by the vector $\partial\mathbf{r}/\partial y$, and as a consequence the vector $\partial\mathbf{r}/\partial y$ defines the cross-section of the beam. In the absolute nodal co-ordinate formulation, the global position vector of an arbitrary point on the beam can be written as

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \mathbf{Se}, \tag{3}$$

where $\mathbf{S}$ is the global element shape function, and $\mathbf{e}$ is the vector of nodal co-ordinates. The vector of the element nodal co-ordinates $\mathbf{e}$ is given by

$$\mathbf{e} = [e_1\ \ e_2\ \ e_3\ \ e_4\ \ e_5\ \ e_6\ \ e_7\ \ e_8\ \ e_9\ \ e_{10}\ \ e_{11}\ \ e_{12}]^{\mathrm{T}}. \tag{4}$$

The vector of nodal co-ordinates includes the global displacements

$$e_1 = r_1|_{x=0}, \qquad e_2 = r_2|_{x=0}, \qquad e_7 = r_1|_{x=l}, \qquad e_8 = r_2|_{x=l}$$

and the global slopes of the element nodes that are defined as

$$e_3 = \left.\frac{\partial r_1}{\partial x}\right|_{x=0}, \qquad e_4 = \left.\frac{\partial r_2}{\partial x}\right|_{x=0}, \qquad e_5 = \left.\frac{\partial r_1}{\partial y}\right|_{x=0}, \qquad e_6 = \left.\frac{\partial r_2}{\partial y}\right|_{x=0},$$

$$e_9 = \left.\frac{\partial r_1}{\partial x}\right|_{x=l}, \qquad e_{10} = \left.\frac{\partial r_2}{\partial x}\right|_{x=l}, \qquad e_{11} = \left.\frac{\partial r_1}{\partial y}\right|_{x=l}, \qquad e_{12} = \left.\frac{\partial r_2}{\partial y}\right|_{x=l}.$$

The element shape function $\mathbf{S}$ must have a complete set of rigid body modes that describe arbitrary rigid body translation and rotational displacements. Using the element nodal co-ordinates given by equation (4), the element shape function can be defined as

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 & ls_2 & 0 & ls_3 & 0 & s_4 & 0 & ls_5 & 0 & ls_6 & 0 \\ 0 & s_1 & 0 & ls_2 & 0 & ls_3 & 0 & s_4 & 0 & ls_5 & 0 & ls_6 \end{bmatrix}, \tag{5}$$

where the functions $s_i = s(\xi, \eta)$ are defined as

$$s_1 = 1 - 3\xi^3 + 2\xi^3, \qquad s_2 = \xi - 2\eta^2 + \xi^3, \qquad s_3 = \eta - \xi\eta,$$

$$s_4 = 3\xi^2 - 2\xi^3, \qquad s_5 = -\,\xi^2 + l\xi^3, \qquad s_6 = \xi\eta$$

and $\xi = x/l$, $\eta = y/l$; l is the element length.

3. CONSTANT MASS MATRIX

The global position vector of an arbitrary point on the shear-deformable beam is given by equation (3). By differentiating this equation with respect to time, the absolute velocity vector can be defined as $\dot{\mathbf{r}} = \mathbf{S}\dot{\mathbf{e}}$. This vector can be used to define the kinetic energy of the element as

$$T = \frac{1}{2}\int_V \rho\dot{\mathbf{r}}^{\mathrm{T}}\,\dot{\mathbf{r}}\,\mathrm{d}V = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\left(\int_V \rho\mathbf{S}^{\mathrm{T}}\mathbf{S}\,\mathrm{d}V\right)\dot{\mathbf{e}} = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{M}_a\dot{\mathbf{e}}, \tag{6}$$

where V is the volume, ρ the mass density of the beam material, and $\mathbf{M}_a$ the mass matrix of the element. The mass matrix in equation (6) is constant and symmetric. Using the shape function given by equation (5), the mass matrix is given by

$$\mathbf{M}_a = \int_V \rho \mathbf{S}^{\mathrm{T}}\mathbf{S}\,\mathrm{d}V =$$

$$\begin{bmatrix}
\frac{13}{35}m & 0 & \frac{11}{210}ml & 0 & \frac{7}{20}J_1 & 0 & \frac{9}{70}m & 0 & \frac{-13}{420}ml & 0 & \frac{3}{20}J_1 & 0 \\
0 & \frac{13}{35}m & 0 & \frac{11}{210}ml & 0 & \frac{7}{20}J_1 & 0 & \frac{9}{70}m & 0 & \frac{-13}{420}ml & 0 & \frac{3}{20}J_1 \\
\frac{11}{210}ml & 0 & \frac{1}{105}ml^2 & 0 & \frac{1}{20}J_1l & 0 & \frac{13}{420}ml & 0 & \frac{-1}{140}ml^2 & 0 & \frac{1}{30}J_1l & 0 \\
0 & \frac{11}{210}ml & 0 & \frac{1}{105}ml^2 & 0 & \frac{1}{20}J_1l & 0 & \frac{13}{420}ml & 0 & \frac{-1}{140}ml^2 & 0 & \frac{1}{30}J_1l \\
\frac{7}{20}J_1 & 0 & \frac{1}{20}J_1l & 0 & \frac{1}{3}J_2 & 0 & \frac{3}{20}J_1 & 0 & \frac{-1}{30}J_1l & 0 & \frac{1}{6}J_1 & 0 \\
0 & \frac{7}{20}J_1 & 0 & \frac{1}{20}J_1l & 0 & \frac{1}{3}J_2 & 0 & \frac{3}{20}J_1 & 0 & \frac{-1}{30}J_1l & 0 & \frac{1}{6}J_2 \\
\frac{9}{70}m & 0 & \frac{13}{420}ml & 0 & \frac{3}{20}J_1 & 0 & \frac{13}{35}m & 0 & \frac{-11}{210}ml & 0 & \frac{7}{20}J_1 & 0 \\
0 & \frac{9}{70}m & 0 & \frac{13}{420}ml & 0 & \frac{3}{20}J_1 & 0 & \frac{13}{35}m & 0 & \frac{-11}{210}ml & 0 & \frac{7}{20}J_1 \\
\frac{-13}{420}ml & 0 & \frac{-1}{140}ml^2 & 0 & \frac{-1}{30}J_1l & 0 & \frac{-1}{210}ml & 0 & \frac{1}{105}ml^2 & 0 & \frac{-1}{20}J_1l & 0 \\
0 & \frac{-13}{420}ml & 0 & \frac{-1}{140}ml^2 & 0 & \frac{-1}{30}J_1l & 0 & \frac{-11}{210}ml & 0 & \frac{1}{105}ml^2 & 0 & \frac{-1}{20}J_1l \\
\frac{3}{20}J_1 & 0 & \frac{1}{30}J_1l & 0 & \frac{1}{6}J_2 & 0 & \frac{7}{20}J_1 & 0 & \frac{-1}{20}J_1l & 0 & \frac{1}{3}J_2 & 0 \\
0 & \frac{3}{20}J_1 & 0 & \frac{1}{30}J_1l & 0 & \frac{1}{6}J_2 & 0 & \frac{7}{20}J_1 & 0 & \frac{-1}{20}J_1l & 0 & \frac{1}{3}J_2
\end{bmatrix}, \quad (7)$$

where m is the total mass of the finite element, l the element length, J_1 the first moment of mass defined as $J_1 = \int_V \rho y\,\mathrm{d}V$, and J_2 the second moment of mass defined as $J_2 = \int_V \rho y^2\,\mathrm{d}V$.

4. ELASTIC FORCES

In this section, the non-linear strain–displacement relations are used to develop an expression for the elastic forces of the beam element. The deformation gradient can be defined as [11]

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\begin{bmatrix} \dfrac{\partial r_1}{\partial x} & \dfrac{\partial r_1}{\partial y} \\ \dfrac{\partial r_2}{\partial x} & \dfrac{\partial r_2}{\partial y} \end{bmatrix}\mathbf{J}_0^{-1} = \begin{bmatrix} \mathbf{S}_{1x}\mathbf{e} & \mathbf{S}_{1y}\mathbf{e} \\ \mathbf{S}_{2x}\mathbf{e} & \mathbf{S}_{2y}\mathbf{e} \end{bmatrix}\mathbf{J}_0^{-1}, \quad (8)$$

where $\mathbf{S}_{ix} = \partial \mathbf{S}_i/\partial x$, $\mathbf{S}_{iy} = \partial \mathbf{S}_i/\partial y$, $\mathbf{S}_i$ is the ith row of the element shape function, $\mathbf{J}_0 = \partial \mathbf{X}/\partial \mathbf{x}$ and $\mathbf{X} = \mathbf{S}\mathbf{e}_0$ where $\mathbf{e}_0$ is the vector of absolute nodal coordinates in the initial configuration. The matrix $\mathbf{J}_0$ must be considered in formulating the elastic forces if the beam has an arbitrary initial configuration. Note that $\mathbf{J}_0$ is the identity matrix for initially horizontal beams. In the following discussion, we consider the simple case of a beam that is horizontal in the initial configuration. The Lagrangian strain tensor ε_m in the case of initially horizontal beam can be written as

$$\varepsilon_m = \frac{1}{2}(\mathbf{J}^{\mathrm{T}}\mathbf{J} - \mathbf{I}) = \frac{1}{2}\begin{bmatrix} \mathbf{e}^{\mathrm{T}}\mathbf{S}_a\mathbf{e} - 1 & \mathbf{e}^{\mathrm{T}}\mathbf{S}_c\mathbf{e} \\ \mathbf{e}^{\mathrm{T}}\mathbf{S}_c\mathbf{e} & \mathbf{e}^{\mathrm{T}}\mathbf{S}_b\mathbf{e} - 1 \end{bmatrix}, \tag{9}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{S}_a = \mathbf{S}_{1x}^{\mathrm{T}}\mathbf{S}_{1x} + \mathbf{S}_{2x}^{\mathrm{T}}\mathbf{S}_{2x}$, $\mathbf{S}_b = \mathbf{S}_{1y}^{\mathrm{T}}\mathbf{S}_{1y} + \mathbf{S}_{2y}^{\mathrm{T}}\mathbf{S}_{2y}$, and $\mathbf{S}_c = \mathbf{S}_{1x}^{\mathrm{T}}\mathbf{S}_{1y} + \mathbf{S}_{2x}^{\mathrm{T}}\mathbf{S}_{2y}$. It should be noted that the strain tensor is symmetric; thus its components can be written in a vector form as

$$\varepsilon = [\varepsilon_1 \; \varepsilon_2 \; \varepsilon_3]^{\mathrm{T}}, \tag{10}$$

where $\varepsilon_1 = \frac{1}{2}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_a\mathbf{e} - 1)$, $\varepsilon_2 = \frac{1}{2}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_b\mathbf{e} - 1)$, and $\varepsilon_3 = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{S}_c\mathbf{e}$.

A general expression for the strain energy can be written using the strain vector ε and the stress vector $\sigma = [\sigma_1 \; \sigma_2 \; \sigma_3]^{\mathrm{T}}$ as follows [12, 13]:

$$U = \frac{1}{2}\int_V \sigma^{\mathrm{T}}\varepsilon \, \mathrm{d}V. \tag{11}$$

Using the constitutive equations, the stress vector is related to the strain vector by

$$\sigma = \mathbf{E}\varepsilon, \tag{12}$$

where $\mathbf{E}$ is the matrix of the elastic constants of the material. For isotropic homogenous material, matrix $\mathbf{E}$ can be expressed in terms of Lame's constants λ and μ as

$$\mathbf{E} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix}, \tag{13}$$

where $\lambda = Ev/[(1 + v)(1 - 2v)]$, $\mu = E/[2(1 + v)]$, E is Young's modulus of elasticity, and v is the Poisson's ratio of the beam material. Using equation (12) and (13), the strain energy can be rewritten as

$$U = \frac{1}{2}\int_V \varepsilon^{\mathrm{T}}\mathbf{E}\varepsilon \, \mathrm{d}V. \tag{14}$$

The vector of the elastic forces $\mathbf{Q}_e$ can be defined using the strain energy U as

$$\mathbf{Q}_e^{\mathrm{T}} = \frac{\partial U}{\partial \mathbf{e}} = \mathbf{e}^{\mathrm{T}}\mathbf{K}, \tag{15}$$

where $\mathbf{K}$ is the stiffness matrix which can be written as

$$\mathbf{K} = (\lambda + 2\mu)\mathbf{K}_1 + \lambda\mathbf{K}_2 + \mathbf{2}\mu\mathbf{K}_3, \tag{16}$$

where

$$\mathbf{K}_1 = \frac{1}{4}\int_V [\mathbf{S}_{a1}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_a\mathbf{e} - 1) + \mathbf{S}_{b1}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_b\mathbf{e} - 1)] \, \mathrm{d}V,$$

$$\mathbf{K}_2 = \frac{1}{4}\int_V [\mathbf{S}_{a1}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_a\mathbf{e} - 1) + \mathbf{S}_{b1}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_a\mathbf{e} - 1)] \, \mathrm{d}V, \qquad \mathbf{K}_3 = \frac{1}{4}\int_V [\mathbf{S}_{c1}(\mathbf{e}^{\mathrm{T}}\mathbf{S}_a\mathbf{e}) \, \mathrm{d}V,$$

$$\mathbf{S}_{a1} = \mathbf{S}_a + \mathbf{S}_a^{\mathrm{T}}, \qquad \mathbf{S}_{b1} = \mathbf{S}_b + \mathbf{S}_b^{\mathrm{T}}, \qquad \text{and } \mathbf{S}_{c1} = \mathbf{S}_c + \mathbf{S}_c^{\mathrm{T}}.$$

Note that the general expression for the elastic forces obtained in this investigation using the continuum mechanics approach and the non-linear strain–displacement relationships is much simpler than the expression obtained in previous investigations [5, 7] using the element local co-ordinate system and linear strain–displacement relationships. The general expression obtained in this paper automatically captures the effect of geometric centrifugal stiffening.

5. FORMULATION OF THE GENERALIZED EXTERNAL FORCES

The virtual work can be used to develop the vector of the generalized external forces [4]. The virtual work due to an externally applied force $\mathbf{F}$ acting on an arbitrary point on the element is given by

$$\delta W_e = \mathbf{F}^{\mathrm{T}}\delta\mathbf{r} = \mathbf{F}^{\mathrm{T}}\mathbf{S}\delta\mathbf{e} = \mathbf{Q}_e^{\mathrm{T}}\delta\mathbf{e}, \tag{17}$$

where $\mathbf{r}$ is the position vector of the point of application of the force, and $\mathbf{Q}_e = \mathbf{S}^{\mathrm{T}}\mathbf{F}$ is the vector of the generalized forces associated with the element nodal co-ordinates. For example, the virtual work due to the distributed gravity of the finite element can be obtained using the shape function and equation (17) as

$$\int_V [0 \quad -\rho g]\,\mathbf{S}\delta\mathbf{e} = -\,mg\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{l}{12} & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{l}{12} & 0 & 0 \end{bmatrix}\delta\mathbf{e}. \tag{18}$$

This defines the vector of generalized distributed gravity forces as

$$\mathbf{Q}_e = -\,mg\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{l}{12} & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{l}{12} & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{19}$$

When an external moment M acts on the cross-section of the beam, the virtual work due to this moment is given by

$$\delta W_M = M\delta\gamma, \tag{20}$$

where γ is the angle of rotation of the cross-section. The orientation of a co-ordinate system attached to the cross-section can be defined using the transformation matrix

$$\begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} = \frac{1}{f}\begin{bmatrix} \dfrac{\partial r_2}{\partial y} & \dfrac{\partial r_1}{\partial y} \\ -\dfrac{\partial r_1}{\partial y} & \dfrac{\partial r_2}{\partial y} \end{bmatrix}, \qquad f = \sqrt{\left(\frac{\partial r_1}{\partial y}\right)^2 + \left(\frac{\partial r_2}{\partial y}\right)^2}, \tag{21}$$

which yields

$$\cos\gamma = \frac{1}{f}\frac{\partial r_2}{\partial y}, \qquad \sin\gamma = -\frac{1}{f}\frac{\partial r_1}{\partial y}. \tag{22}$$

Using these two equations, it can be shown that the virtual change in the cross-section orientation angle can be defined as

$$\delta\gamma = \frac{(\partial r_2/\partial y)\,\delta(\partial r_1/\partial y) - (\partial r_1/\partial y)\,\delta(\cdot r_2/\partial y)}{f^2}. \tag{23}$$

If a concentrated moment is applied for example at node A, the generalized force vector due to this moment is defined as

$$\mathbf{Q}_M = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{Me_6}{f_A^2} & \dfrac{-Me_5}{f_A^2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{24}$$

6. NUMERICAL EXAMPLES

Using the obtained expressions of the elastic and inertia forces, the equations of motion can be obtained using the absolute nodal co-ordinates as

$$\mathbf{M}_a \ddot{\mathbf{e}} = \mathbf{Q}, \tag{25}$$

where $\mathbf{Q}$ is the vector of generalized nodal forces including the elastic force vector $\mathbf{Q}_e$ and external force vector $\mathbf{Q}_k$, and $\ddot{\mathbf{e}}$ is the vector of absolute accelerations. As previously mentioned, the centrifugal and Coriolis force vectors are equal to zero since the mass matrix is constant.

In this section, an example is considered in order to demonstrate the performance of the proposed beam model. The example considered is the free falling of a flexible pendulum under its own weight. The free falling two-dimensional beam is shown in Figure 3. The beam is connected to the ground by a pin joint at one end. The beam has length of 1·2 m, circular cross-sectional area of 0·0016 m^2, second moment area of 8·533E-06, a mass density of 5540 kg/m^3, Poisson's ratio of 0·3, and a modulus of elasticity of 0·700E 06. The beam is initially horizontal with zero initial velocity and free to fall under the effect of gravity. The gravity constant is assumed to be 9·81 m/s^2. The simulations of the beam are performed using different numbers of elements. Figure 4 shows the position of the tip point of the beam using 6 and 12 finite elements when the gravity constant is equal to 9·81 m/s^2. It is clear from the results presented in this figure that there is a good agreement between the two models. The results demonstrate that the solution converges with small numbers of elements.

Since the falling pendulum is a conservative system, the sum of the energy components must remain constant, that is

$$\sum_i^{ne} T^i + U^i + V^i = const, \tag{26}$$

where T^i is the element kinetic energy, U^i is the element strain energy, V^i is the element potential energy, and ne is the number of elements of the system. Figure 5 shows the energy balance for a six-element model. The results presented in this figure show that the total energy remains constant. The results obtained using the six-element model are the same as

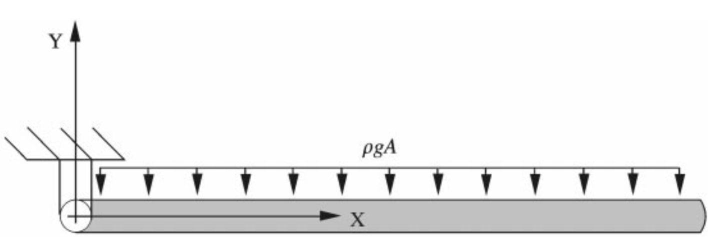

Figure 3. Free falling flexible pendulum.

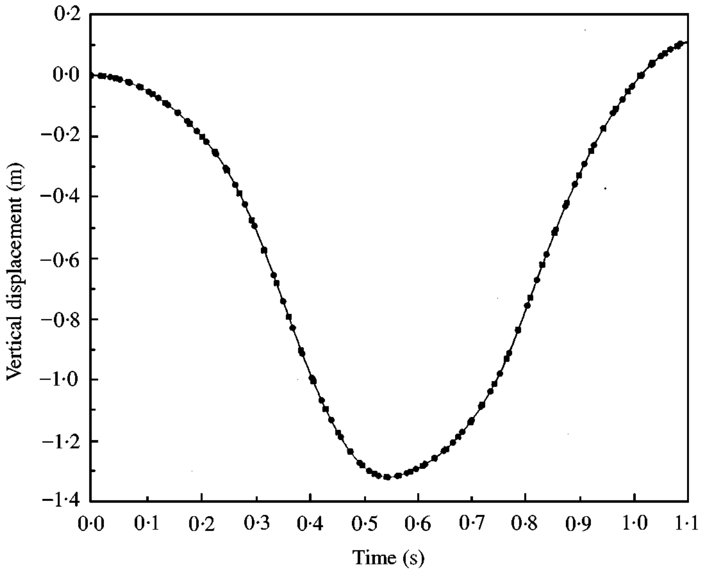

Figure 4. Displacement of the beam tip point under acceleration 9·81 m/s^2 using: ---■---, 6 elements; —●—, 12 elements.

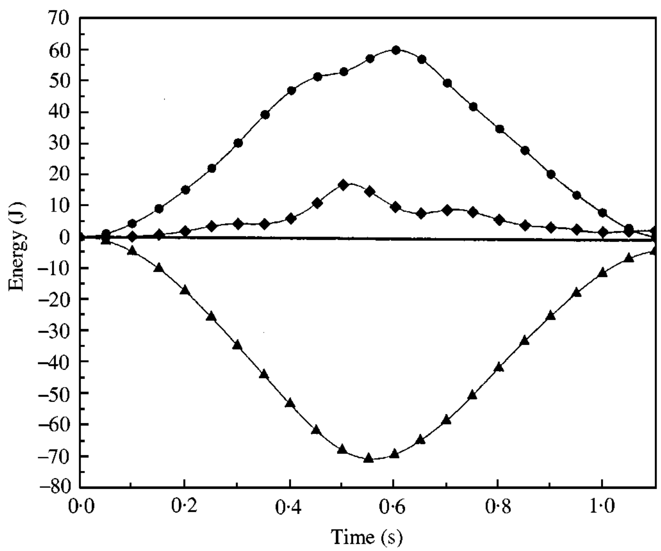

Figure 5. Energy balance of the beam ($g = 9{\cdot}81$ m/s^2): —●—, kinetic energy; —▲—, potential energy; ---◆---, elastic energy; ▬▬, total energy.

the results of the 12-element model. Figure 6 shows the large deformation configurations of the falling beam at different time steps under gravitational acceleration of 50 m/s^2 using 12 finite elements.

The shear-deformable model developed in this investigation relaxes the assumption of Euler–Bernoulli beam theory. In this model, the cross-section does not remain perpendicular to the beam centerline due to the effect of the shear deformation. Figure 7

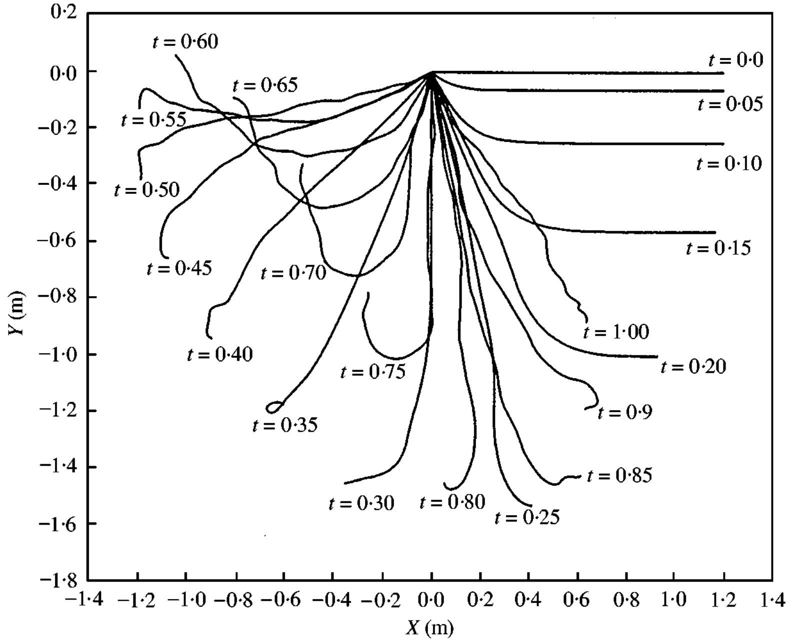

Figure 6. The falling flexible pendulum at different time steps using 12 elements ($g = 50$ m/s^2).

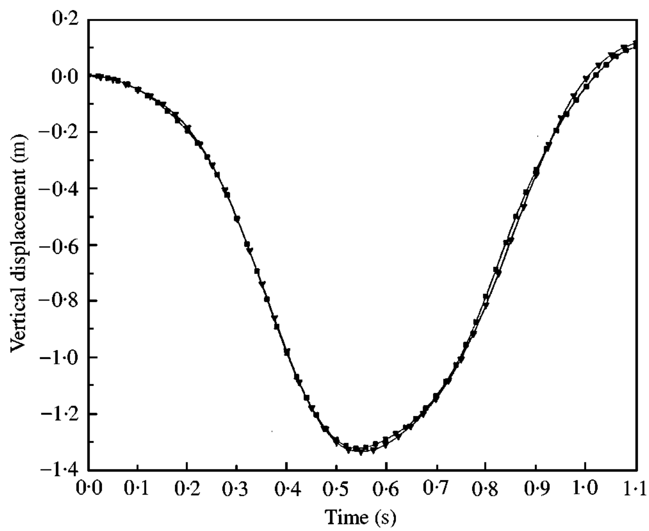

Figure 7. Comparison between the Euler–Bernoulli beam and shear-deformable beam: —▼—, Euler–Bernoulli beam model; —■—, shear-deformable beam.

shows a comparison between the results obtained using this new model and the Euler–Bernoulli beam model previously presented by Berzeri and Shabana [9] who used a shape function that does not depend on y. The results in this figure are obtained using 12 elements. It is important to point out that the shear-deformable model, because of the dependence of the shape function on y, leads to a simpler expression for the elastic forces.

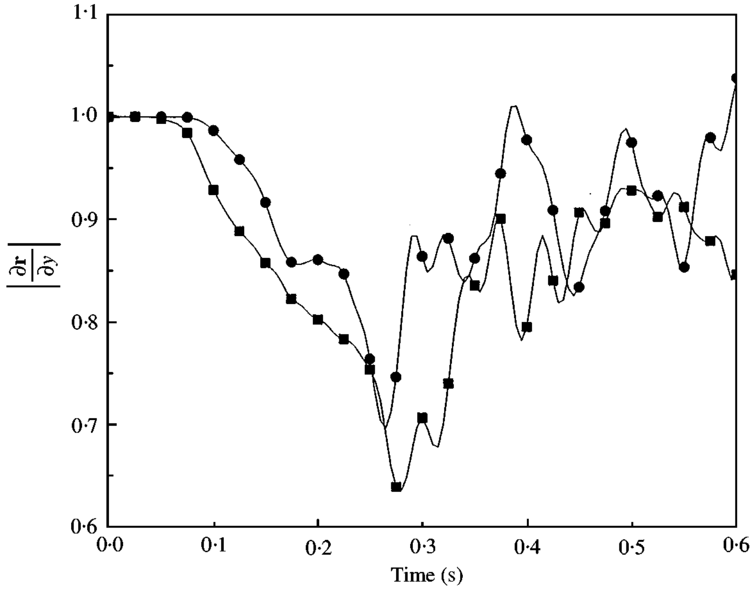

Figure 8. The norm of the vector $\partial \mathbf{r}/\partial y$ as function of time at: —■—, Node 5; —●—, Node 9.

More surprisingly, significant saving in computer time was achieved using the shear-deformable model. It was observed that the shear-deformable model is two times faster than the Euler–Bernoulli model. The results presented in Figure 7 show that there is a very good agreement between the shear-deformable beam model and the model based on the Euler–Bernoulli beam theory since a thin beam is used.

The shear-deformable model developed in this investigation also relaxes the assumption of some Timoshenko beam models since it allows for the plane deformation of the cross-section, that is the cross-section remains plane but not rigid. The deformation of the cross-section can be measured by the deviation of the norm of the vector $\partial \mathbf{r}/\partial y$ from one. Figure 8 shows the norm of this vector at node 5 and node 9 as a function of time in the case of free falling pendulum and gravity constant of 50 m/s^2.

7. SUMMARY AND CONCLUSIONS

In this investigation, a shear-deformable beam model based on the non-incremental absolute nodal co-ordinate formulation is developed for the large rotation and large deformation analysis. The beam model leads to exact modelling of the rigid body dynamics and leads to zero strain under an arbitrary rigid body displacement. Furthermore, the model relaxes some of the assumptions of Euler–Bernoulli and Timoshenko beam models. The cross-section in the new model does not remain rigid and does not remain perpendicular to the beam centerline. By using a continuum mechanics approach, the model leads to a relatively simple expression for the elastic forces.

While the model accounts for the effect of the rotary inertia and shear deformation, the mass matrix remains constant. As a consequence, the centrifugal and Coriolis forces are identically equal to zero. A numerical example, a free falling pendulum, was used to demonstrate the use of the new beam model. Numerical results obtained in this study demonstrated that the solutions obtained using the new model converges much faster as compared to the non-shear-deformable models presented in previous investigations. Furthermore, significant saving in computer time is achieved by using the more general shear-deformable model that is based on the non-linear strain-displacement relationships.

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